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Leader Group Selection for Herdability of Structurally Balanced Signed Networks

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***Abstract*—This work studies the herdability of networked systems. As an extension of classical controllability, herdability represents the ability of a system to drive its states to a specific subset in the state space. Particularly, a leader-follower signed network is considered, where leader nodes endowed with external controls are able to influence the states of the follower nodes. Weighted positive and negative edges are allowed to capture cooperative and competitive interactions among nodes. To enable network herdability, leader group selection is investigated in this work, i.e., identifying a small subset of nodes as leaders such that the resulting leader-follower network is herdable by the selected leaders. Focusing on structurally balanced signed networks, graph walks are leveraged to facilitate leader selection for network herdability. The cases of selecting leaders from the same partitioned set and different sets are considered, which are then extended to a special class of weakly balanced signed graphs. Examples are provided to illustrate the developed leader selection approaches.**

I. Introduction

Controllability of leader-follower networks, i.e., the capability of driving followers' states by leaders via external controls, has attracted growing research interest in multi-agent systems [1], brain networks [2], and social networks [3]. Among these applications, most existing works focus on ensuring fully controllable networks. That is, all followers' states can be driven to arbitrary states by the selected leaders. However, requiring a network to be fully controllable can be restrictive and unnecessary in practical applications. For example, when controlling the humidity of rooms in a smart building, the rooms are often required to maintain a positive humidity. A fully controllable network is unnecessary in this case, since driving the room humidity (i.e., system states) to a negative value does not make any physical sense. Instead, it is of more practical significance to consider relaxed controllability, i.e., the capability of driving followers' states to a specific subset, rather than the entire state space as in classical controllability. Such a relaxed controllability is referred to as herdability and this work is particularly motivated to develop leader selection algorithms for the herdability of leader-follower networks.

Based on the interactions, networked systems can be classified as either cooperative or non-cooperative networks. Cooperative networks are commonly modeled by unsigned graphs (i.e., graphs with only positive edge weights), where positive weights indicate cooperative relationships between network units. Average consensus is an example of cooperative networks, where agents collaborate to achieve group consensus [4]. Non-cooperative networks are often modeled by signed graphs, which admit both positive and negative edge weights to represent cooperative and antagonistic interactions [5]. For instance, signed graphs with positive/negative weights can be used to model friend/adversary relationship in social networks [6] and collaborative/competitive relationship in multi-agent systems [7]. Owing to tremendous application potential of signed graphs, this work is further motivated to investigate the herdability of leader-follower signed networks.

Investigating leader group selection for the controllability of unsigned networks has generated a substantial research volume. Structural controllability [8]–[10], graph-theoretic approaches [11]–[13], and topological properties [14]–[16], were extensively explored to facilitate leader group selection to ensure the controllability of unsigned networks. When considering signed networks, the controllability was studied in the works of [17]–[20], and the leader selection algorithms ensuring network controllability were investigated in the works of [21]–[23]. However, all of the aforementioned results focus on the characterization of classical controllability. Leader group selection for network herdability remains largely unknown.

The idea of herdability was introduced in [24], in which necessary and sufficient conditions for herdable systems were presented. The results of [24] were then extended to characterize herdability for positive complex networks in [25] and signed linear systems in [26]. In our recent works of [27] and [28], herdability of signed networks was characterized from topological perspectives. Inspired by the aforementioned works of [24]–[28], this work advances current knowledge by considering leader group selection for network herdability. That is, given a signed network, we are concerned with identifying a small subset of nodes as leaders, such that the resulting leader-follower network is herdable by the selected leaders. Specifically, since graph walks are closely related to the controllable space of the system, graph walks are exploited to facilitate leader selection for the herdability of structurally balanced signed networks. The cases of selecting leaders from the same partitioned set and different sets are considered, which are then extended to a special class of weakly balanced signed networks. Examples are provided to elaborate the developed leader selection approaches. Compared with existing literature, this

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work is one of the first attempts to consider leader selection for network herdability. Although only structurally balanced networks are investigated, it paves the way to leader selection for more general signed networks.

II. Problem Formulation

Consider a network modeled by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set. A graph is connected, if there exists a series of consecutive edges connecting any two nodes. The interactions between nodes are described by the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0, \forall i = 1, \ldots, n$. Let $\mathbb{R}^+$ and $\mathbb{R}^-$ represent the set of positive and negative real numbers, respectively. In this work, the weight $a_{ij} \in \mathbb{R}$ is allowed to take real numbers, where $a_{ij} \in \mathbb{R}^+$ and $a_{ij} \in \mathbb{R}^-$ represent cooperative and competitive interactions between v_i and v_j, respectively. Throughout the rest of this work, an edge (v_j, v_i) is called positive if $a_{ij} \in \mathbb{R}^+$, and negative otherwise. The ith row and jth column of $\mathcal{A}$ are denoted by $\mathcal{A}_{i,:}$ and $\mathcal{A}_{:,j}$, respectively.

A. Network Herdability

Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the stacked system states[1] of the network $\mathcal{G}$, where each entry $x_i(t) \in \mathbb{R}$ represents the state of node v_i. It is assumed that a subset $\mathcal{V}_l \subseteq \mathcal{V}$ of m nodes, referred to as leaders in the network, can be endowed with external controls. The rest nodes $\mathcal{V}_f = \mathcal{V}\backslash\mathcal{V}_l$ are referred to as followers with $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$. Without loss of generality, the leaders' and the followers' indices are assumed to be $\mathcal{V}_l = \{1, \ldots, m\}$ and $\mathcal{V}_f = \{m + 1, \ldots, n\}$.

Since linear dynamics dependent on the adjacency matrix of the underlying graph has found many applications in [29]–[32], this work considers

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t), \tag{1}$$

where $\mathcal{A}$ is the adjacency matrix of $\mathcal{G}$, $B = [e_1 \cdots e_m] \in \mathbb{R}^{n\times m}$ is the input matrix with basis vectors e_i, $i = 1, \ldots, m$, indicating that the ith node is a leader, and $u(t) \in \mathbb{R}^m$ is the external control.

The herdability of the system in (1) is defined as follows.

Definition 1 (Network Herdability [24])**.** A networked system with dynamics in (1) is herdable if, for any $x(0) \in \mathbb{R}^n$, the system state $x(t)$ can be driven by a control input $u(t)$ to the set $H_d = \left\{x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \in \mathbb{R}^n : x_i \geq d\right\}$ in finite time, where d is an arbitrary positive threshold.

Definition 1 indicates that a network is herdable if its states can be driven to a specific subset H_d of the state space. Recall that the controllability matrix $C = \begin{bmatrix} B & \mathcal{A}B & \cdots & \mathcal{A}^{n-1}B \end{bmatrix}$ indicates the controllable subspace of a system. If C has full row rank, the system in (1) is completely controllable. The following lemma shows how the herdability of a system relates to the controllability matrix.

Lemma 1. *[24] A networked system with dynamics in (1) is herdable to H_d if and only if there exists an element-wise positive vector $k \in \operatorname{Im}(C)$, where $\operatorname{Im}(\cdot)$ represents the image space of a matrix.*

Lemma 1 indicates that network herdability depends on the image space of the controllability matrix, which is closely related to the adjacency matrix $\mathcal{A}$ and the input matrix B. Given a network with known adjacency matrix, if a proper set of nodes is selected as leaders (i.e., the input matrix B is well designed), it is possible to drive system states to an arbitrary H_d via external controls. Motivated by this observation, the objective of this work is to develop leader group selection algorithms for ensured network herdability.

B. Graph Walks

In this work, graph walks will be used as a main tool to characterize network herdability and facilitate leader group selection. A graph walk is defined as an alternating sequence of edges and the length of a walk is the number of edges. A walk of length k is referred to as a k-walk. The weight of a k-walk is defined as the product of the edge weights. Different from the commonly used graph path that only contains distinct edges, repeated edges are allowed in graph walks. Thus, the minimum walk between two nodes is the walk of the smallest length.

Consider a weighted signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{A}$ is the weighted adjacency matrix. Let $\mathcal{A}^k$ denote the kth power of $\mathcal{A}$. Each entry of $\mathcal{A}^k$ is determined as $[\mathcal{A}^k]_{ij}$, which is the sum of the weights of all k-walks from v_j to v_i [33]. When considering the product $\mathcal{A}^k B$, the entry $[\mathcal{A}^k B]_{ij}$ indicates the sum of the weights of all k-walks from the leader $v_j \in \mathcal{V}_l$ to a node $v_i \in \mathcal{V}$, which motivates the use of graph walks to characterize the controllable subspace via the controllability matrix C and identify leader groups for network herdability.

III. Herdability of Structurally Balanced Signed Networks

Due to the existence of positive and negative weights, structural balance is a topological feature associated with signed graphs. Exploring structural balance has generated fruitful results regarding the relationships between network controllability and its topological structures [34]. Motivated by this topological feature, we focus on developing leader group selection algorithms for the herdability of structurally balanced signed graphs. In particular, standard structurally balanced graphs are considered first, which will then be extended in Sec. IV to a class of special signed graphs, namely weakly balanced signed graphs.

Definition 2 (Structural Balance [5])**.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if the node set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$,

[1]Generalizations to multi-dimensional system states (e.g., $x_i \in \mathbb{R}^m$) are expected to be trivial via the matrix Kronecker product.

where $a_{ij} > 0$ if $v_i, v_j \in \mathcal{V}_q$, $q \in \{1,2\}$, and $a_{ij} < 0$ if $v_i \in \mathcal{V}_q$ and $v_j \in \mathcal{V}_r$, $q \neq r$, and $q, r \in \{1,2\}$.

Definition 2 indicates that, in structurally balanced graphs, v_i and v_j are positive neighbors if they are from the same partitioned set, and negative neighbors from different sets. Based on Definition 2, leader group selection from one set (i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$) and two sets (i.e., $\mathcal{V}_1$ and $\mathcal{V}_2$) are discussed in Sec. III-A and Sec. III-B, respectively.

A. Selecting Leaders in One Set

When leaders are all in the same partitioned set, systems evolving over structurally balanced graphs have the following properties.

Lemma 2. *[26] If the system in (1) evolves over a connected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and $\mathcal{G}$ is structurally balanced, its controllability matrix C is sign definite.*

Let $C_{i,:}$ denote the ith row of C. Lemma 2 indicates that, for any two nodes $v_p \in \mathcal{V}_1$ and $v_q \in \mathcal{V}_2$, the entries of $C_{p,:}$ are all non-negative and the entries of $C_{q,:}$ are all non-positive, given that the leaders are all in $\mathcal{V}_1$.

Assume $|\mathcal{V}_1| = n_1$ and $|\mathcal{V}_2| = n_2$ with $n_1 + n_2 = n$. An intuitive leader group selection method is presented as follows.

Proposition 1. *Consider a leader-follower system in (1) evolving over a signed graph $\mathcal{G}$. Suppose $\mathcal{G}$ is connected and structurally balanced with nodes grouped into $\mathcal{V}_1$ and $\mathcal{V}_2$ as in Definition 2. If the nodes in either $\mathcal{V}_1$ or $\mathcal{V}_2$ are all selected as leaders, the system in (1) is herdable by the selected leaders.*

Proof: Without loss of generality, assume the nodes in $\mathcal{V}_1$ are all selected as leaders, i.e., there are m leaders where $m = n_1$ and n_2 followers. The controllability matrix C of the system in (1) can be written as

$$C = \left[\begin{array}{c|ccc} B & \mathcal{A}B & \dots & \mathcal{A}^{n-1}B \end{array} \right] = \left[\begin{array}{c|c} I_{m\times m} & * \\ 0_{n_2\times m} & \Xi \end{array} \right], \tag{2}$$

where $* \in \mathbb{R}^{m\times(n-1)m}$ and $\Xi \in \mathbb{R}^{n_2\times(n-1)m}$. Since $\mathcal{G}$ is connected, any two nodes are connected within $(n-1)$-walks. As a result, the entries in $*$ and Ξ represent the weights of walks from leaders to leaders and from leaders to followers, respectively.

Based on Lemma 1, the system in (1) is herdable if and only if there exists an element-wise positive vector $k \in \mathrm{Im}(C)$. That is, there exists a vector $\delta = \left[\begin{array}{ccc} \delta_1 & \dots & \delta_{nm} \end{array} \right]^T \in \mathbb{R}^{nm}$ such that $k = C\delta \in \mathbb{R}^n$ is element-wise positive. The rest of the proof is to show the existence of such a vector δ.

Note that Ξ only contains the weights of walks from leaders to followers. By Lemma 2, the entries of $*$ are non-negative and the entries of Ξ are non-positive. In addition, since the graph is connected and any follower can be reached by a leader via walks, no rows in Ξ are all zeros. Therefore, there exists $\delta_i \in \mathbb{R}^-$, $i = m+1, \dots, nm$, such that the last n_2 entries in k are guaranteed to be positive. Due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \dots, m$, such that $\delta_i > \sum_{j=m+1}^{nm} C_{i,j}\delta_j$, which indicates the first m entries in k are guaranteed to be positive. Consequently, due to the existence of a vector δ, the system in (1) is herdable when the nodes in $\mathcal{V}_1$ are all selected as leaders. ■

Proposition 1 offers a straightforward way to select leaders for the herdability of structurally balanced networks. This is particularly useful if the ratio between $|\mathcal{V}_1|$ and $|\mathcal{V}_2|$ is sufficiently small, since the entire network can be effectively herded by a small set of leaders. However, if the partitioned sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are approximate the same size, Proposition 1 can result in a large number of leaders.

Remark 1. Let the nodes in $\mathcal{V}_1$ with 1-walk neighbors in $\mathcal{V}_2$ be referred to as boundary nodes of $\mathcal{V}_1$. Boundary leaders are referred to as leaders that are boundary nodes. Following similar proof of Proposition 1, it can be shown that the system in (1) remains herdable, if the selected leader group satisfies that the followers in $\mathcal{V}_1$ are all within 1-walk from at least one non-boundary leader.

The following theorem presents how the leader selection algorithm in Proposition 1 can be improved by reducing the leader group size based on graph distances between followers and leaders. Suppose there are $m < n_1$ leaders in $\mathcal{V}_1$ and the rest nodes are followers in $\mathcal{V}_1$ and $\mathcal{V}_2$. Let P_{ij}^a, $i \in \{1, \dots, m\}$, $j \in \{m+1, \dots, n\}$, denote the distance (i.e., the length of the minimum walk) from a leader v_i to a follower v_j in $\mathcal{V}_a$, $\{a\} \in \{1,2\}$. The shortest distance from a follower v_j in $\mathcal{V}_a$ to the leader group is denoted by $\underline{P}_j^a = \min\left\{P_{ij}^a, i \in \{1, \dots, m\}\right\}$. Similarly, let $\bar{P}_j^a = \max\left\{P_{ij}^a, i \in \{1, \dots, m\}\right\}$ denote the longest distance from a follower v_j in $\mathcal{V}_a$ to the leader group.

Theorem 1. *Consider a leader-follower system in (1) evolving over a connected structurally balanced signed graph $\mathcal{G}$ with nodes grouped into $\mathcal{V}_1$ and $\mathcal{V}_2$. Suppose there are $m < n_1$ leaders in $\mathcal{V}_1$. The system in (1) is herdable if, for any followers $v_i \in \mathcal{V}_1$ and $v_j \in \mathcal{V}_2$, the selected leaders satisfy $\bar{P}_i^1 < \underline{P}_j^2$.*

Proof: Following Proposition 1, assume the leaders are only selected from $\mathcal{V}_1$. Let $\bar{P}^1 = \max\left\{\bar{P}_i^1, i \in \{m+1, \dots, n_1\}\right\}$ denote the maximum distance (in terms of walks) from followers to the leader group in $\mathcal{V}_1$. Given m leaders in $\mathcal{V}_1$ satisfying the proposed rule, the system controllability matrix C can be written as

$$\begin{aligned} C &= \left[\begin{array}{c|ccc|cc} B & \mathcal{A}B & \cdots & \mathcal{A}^{\bar{P}^1}B & \dots & \mathcal{A}^{n-1}B \end{array} \right] \\ &= \left[\begin{array}{c|c|c} I_{m\times m} & *_{m\times m\bar{P}^1} & \dots \\ 0_{(n_1-m)\times m} & \Lambda & \dots \\ 0_{n_2\times m} & 0_{n_2\times m\bar{P}^1} & \Delta \end{array} \right], \end{aligned} \tag{3}$$

where $\Lambda \in \mathbb{R}^{(n_1-m)\times m\bar{P}^1}$ and $\Delta \in \mathbb{R}^{n_2\times m\left(n-1-\bar{P}^1\right)}$. Similar to Proposition 1, the rest of the proof is to show the existence of a vector $\delta = \left[\begin{array}{ccc} \delta_1 & \dots & \delta_{nm} \end{array} \right]^T \in \mathbb{R}^{nm}$

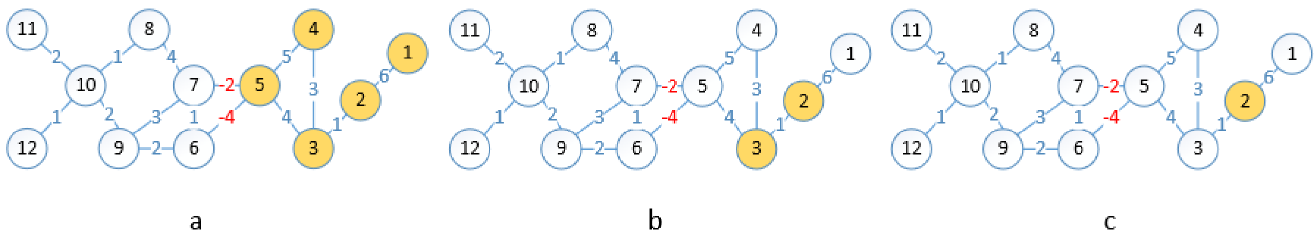

Figure 1. Examples of leader group selection in one set for network herdability, where the selected leaders are marked as solid nodes. Figures (a), (b), and (c) follow the developed leader selection algorithms in Proposition 1, Remark 1 and Theorem 1 respectively.

such that $k = C\delta \in \mathbb{R}^n$ is element-wise positive.

First, due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \ldots, m$, such that the first m entries in k are guaranteed to be positive.

Since the entries of Λ represent the sum of weight within $\bar{P}^1$ walks from leaders to the $n_1 - m$ followers in $\mathcal{V}_1$, these entries are guaranteed to be non-negative, as the nodes in $\mathcal{V}_1$ are positive neighbors. Clearly, there exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = m+1, \ldots, m\left(\bar{P}^1 + 1\right)$, such that the corresponding entries in k are guaranteed to be positive. In addition, if the selected leaders satisfy $\bar{P}_i^1 < \underline{P}_j^2$ for any followers $v_i \in \mathcal{V}_1$ and $v_j \in \mathcal{V}_2$, no followers in $\mathcal{V}_2$ are within $\bar{P}^1$ walks to the leader group. As a result, one has the zero matrix $0_{n_2 \times m\bar{P}^1}$.

By Lemma 2, the entries of Δ are non-positive, since they correspond to the weights of walks from leaders to the followers in $\mathcal{V}_2$. There always exist $\delta_i \in \mathbb{R}^-$, $i = m\left(\bar{P}^1 + 1\right) + 1, \ldots, mn$, such that the last n_2 entries of k are positive. Therefore, due to the existence of a positive element-wise vector $k \in \operatorname{Im}(C)$, the system is herdable based on Lemma 1. ■

Theorem 1 only requires that the longest distance from the followers in $\mathcal{V}_1$ to the leader group is less than the shortest distance from the followers in $\mathcal{V}_2$ to the leader group. Note that the walks among nodes can be computed first using existing algorithms [33]. The leaders can then be selected following the condition in Theorem 1.

To illustrate the leader selection rules above, the following example is provided.

Example 1. Consider a structurally balanced signed network $\mathcal{G}$ with the partitioned sets $\mathcal{V}_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $\mathcal{V}_2 = \{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Following the leader selection rules in Proposition 1, Fig. 1 (a) shows a herdable network with all nodes in $\mathcal{V}_1$ selected as the leader group. Fig. 1 (b) and (c) illustrate the leader selection rules developed in Remark 1 and Theorem 1, respectively.

B. Selecting Leaders in Two Sets

This section considers selecting leaders from both $\mathcal{V}_1$ and $\mathcal{V}_2$ in a structurally balanced graph.

Theorem 2. *Consider a leader-follower system in (1) evolving over a connected structurally balanced signed graph $\mathcal{G}$ with nodes grouped into $\mathcal{V}_1$ and $\mathcal{V}_2$. The system in (1) is herdable, if the boundary nodes of $\mathcal{V}_1$ and $\mathcal{V}_2$ are all selected as leaders.*

Proof: Suppose there are m boundary nodes. Assume there are m_1 leaders and q_1 followers in $\mathcal{V}_1$, and m_2 leaders and q_2 followers in $\mathcal{V}_2$, where $m_1 + m_2 = m$. The leader and follower group in $\mathcal{V}_1$ and $\mathcal{V}_2$ are represented as $\mathcal{V}_l^1$, $\mathcal{V}_f^1$, $\mathcal{V}_l^2$, and $\mathcal{V}_f^2$, respectively.

To facilitate the analysis, the followers are grouped based on the distance to the leaders within the same partitioned set. Specifically, the followers in $\mathcal{V}_f^1$ that are k walks away from the leader group $\mathcal{V}_l^1$ are grouped as f_k^1, and the group size is $\left|f_k^1\right|$ such that $\sum_k \left|f_k^1\right| = q_1$. The followers in $\mathcal{V}_f^2$ are defined similarly by f_k^2 and $\left|f_k^2\right|$ where $\sum_k \left|f_k^2\right| = q_2$. Let p and q be the longest distance from followers to leaders within $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively.

Based on the grouped followers, the system controllability matrix C can be written as

$$C = \begin{bmatrix} B & \mathcal{A}B & \ldots & \mathcal{A}^{n-1}B \end{bmatrix}$$
$$= \begin{bmatrix} I_{m\times m} & * & * & \cdots & * & * & * \\ \mathbf{0} & \phi_1^1 & * & \cdots & * & * & * \\ \vdots & \mathbf{0} & \phi_1^2 & \cdots & * & * & * \\ \vdots & \vdots & \mathbf{0} & \ddots & * & * & * \\ \vdots & \vdots & \vdots & \cdots & \phi_p^1 & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \phi_q^2 & * \end{bmatrix}, \tag{4}$$

where $\phi_i^1 \in \mathbb{R}^{\left|f_i^1\right|\times m_1}$, $i = 1, \ldots, p$, represent the sum of weights of all i-walk from $\mathcal{V}_l^1$ to $\mathcal{V}_f^1$, $\phi_j^2 \in \mathbb{R}^{\left|f_j^2\right|\times m_2}$, $j = 1, \ldots, q$, represent the sum of weights of j-walk from $\mathcal{V}_l^2$ to $\mathcal{V}_f^2$, $\mathbf{0}$ represent zero matrices with appropriate dimensions, and $*$ represent matrices of no interest. By Lemma 1, the rest of the proof is to show the existence of a vector $\delta = \begin{bmatrix} \delta_1 & \ldots & \delta_{nm} \end{bmatrix}^T \in \mathbb{R}^{nm}$ such that $k = C\delta \in \mathbb{R}^n$ is element-wise positive.

First, due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \ldots, m$, such that the first m entries in k are guaranteed to be positive. The nonzero entries of ϕ_1^1 indicate that the followers in $f_1^1 \subseteq \mathcal{V}_f^1$ can be reached by the leaders in $\mathcal{V}_l^1$ within 1-walk. Since they are in the same $\mathcal{V}_1$, the nonzero entries of ϕ_1^1 are all positive. Note that the followers in f_i^1, $i > 1$, can not be reached by the leaders in $\mathcal{V}_l^1$ within 1-walk, which gives rise the zero

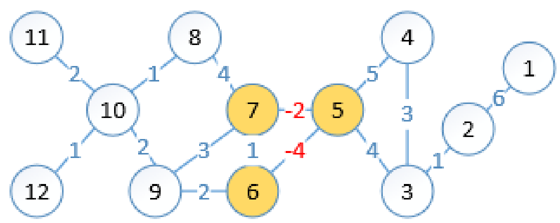

Figure 2. Example of selecting leaders from boundary nodes. The leaders are marked as solid nodes.

matrices. Similarly, the nonzero entries of ϕ_1^2 are positive due to the fact that the followers in $f_1^2 \subseteq \mathcal{V}_f^2$ can be reached by the leaders in $\mathcal{V}_l^2$ within 1-walk and they are in the same $\mathcal{V}_2$. Following similar argument for the rest followers, it can be shown the nonzero entries of ϕ_i^1, $i = 2, \ldots, p$, and ϕ_i^2, $i = 2, \ldots, q$, are all positive. Consequently, there always exist proper design $\delta_i \in \mathbb{R}^+, i = m+1, \ldots mn$, such that k is element-wise positive, which indicates the network $\mathcal{G}$ is herdable. ■

Example 2. Reconsider the structurally balanced signed graph in Example 1. The idea of selecting boundary nodes as leaders is illustrated in Fig. 2.

Remark 2. Theorem 1 and 2 discuss rules of selecting leaders for network herdability. In practice, it is often of interest to modify the network topology by removing or adding edges while preserving network herdability. As an immediate extension of Theorem 1, if new edges are added or existing edges are removed between followers in $\mathcal{V}_2$, the system in (1) remains herdable by the selected leaders as long as the structural balance is preserved. Similarly, as an extension of Theorem 2, adding or removing edges between nodes in the same set will not affect the herdability of the system in (1) by the selected leaders, as long as the structural balance is preserved.

IV. Herdability of Weakly Balanced Networks

This section considers the herdability of a class of special signed graphs, namely weakly balanced signed graphs.

Definition 3 (Weakly Balanced Graphs [35])**.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is weak-balanced if the node set $\mathcal{V}$ can be partitioned such that $\mathcal{V}_1 \cup \mathcal{V}_2 \ldots \cup \mathcal{V}_k = \mathcal{V}$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$, $i \neq j$, and $i, j \in \{1, 2 \ldots, k\}$. The edge weight $a_{ij} > 0$ if $v_i, v_j \in \mathcal{V}_q$, $q \in \{1, 2 \ldots, k\}$, and $a_{ij} < 0$ if $v_i \in \mathcal{V}_q$ and $v_j \in \mathcal{V}_r$, $q \neq r$, and $q, r \in \{1, 2 \ldots, k\}$.

Suppose there are m leaders in total and each $\mathcal{V}_i$, $i \in \{1, 2 \ldots, k\}$, contains a leader group $\mathcal{V}_i' \subset \mathcal{V}_i$ such that $|\mathcal{V}_i'| = m_i$ and $\sum_{i=1}^k m_i = m$. Let P_{ij}^{ab}, $i \in \{1, \ldots, m\}$, $j \in \{m+1, \ldots, n\}$, $a, b \in \{1, 2 \ldots, k\}$, denote the distance from a leader v_i in $\mathcal{V}_a$ to a follower v_j in $\mathcal{V}_b$. The shortest distance from the leader group in $\mathcal{V}_a$ to a follower v_j in $\mathcal{V}_b$ is defined as $\underline{P}_j^{ab} = \min\left\{P_{ij}^{ab}, v_i \in \mathcal{V}_a'\right\}$. Similarly, let $\bar{P}_j^{ab} = \max\left\{P_{ij}^{ab}, v_i \in \mathcal{V}_a'\right\}$ denote the longest distance from the leader group in $\mathcal{V}_a$ to a follower v_j in $\mathcal{V}_b$.

The following theorem presents how the leader group can be identified to ensure herdability of weakly balanced networks.

Theorem 3. *Consider a leader-follower system in (1) evolving over a connected weakly balanced signed graph $\mathcal{G}$. The system in (1) is herdable, if $\bar{P}_j^{aa} < \underline{P}_j^{ba}$, $\forall a, b \in \{1, \ldots, k\}$, is satisfied for any leader set within $\mathcal{V}_b$ and any follower v_j within $\mathcal{V}_a$.*

Proof: For each $\mathcal{V}_a$, $a \in \{1, \ldots, k\}$, suppose there are m_a leaders and f_a followers. Let $\rho_a = \max\left\{\bar{P}_j^{aa}, v_j \in \mathcal{V}_a \setminus \mathcal{V}_a'\right\}$ denote the maximum distance among all followers to the leader group within $\mathcal{V}_a$. The system controllability matrix C can be written as

$$C = \begin{bmatrix} I_{m\times m} & * & * \\ 0_{f_1\times m} & \Lambda_1 & \cdots \\ \vdots & \vdots & \cdots \\ 0_{f_k\times m} & \Lambda_k & \cdots \end{bmatrix}, \tag{5}$$

where $\Lambda_a \in \mathbb{R}^{f_a \times \rho_a m}$ is in the form of

$$\Lambda_a = \begin{bmatrix} \Lambda_{n_a^1\times m} & * & * & * & \cdots \\ 0 & \Lambda_{n_a^2\times m} & * & * & \cdots \\ 0 & 0 & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \Lambda_{n_a^{\rho_a}\times m} & \cdots \end{bmatrix}, \tag{6}$$

where n_a^j, $j \in \{1, \ldots, \rho_a\}$, is the number of followers in $\mathcal{V}_a$ j-walk away from the leader group in $\mathcal{V}_a$. Note that the term $\Lambda_{n_a^1\times m}$ can be further decomposed into $\begin{bmatrix} \Lambda_{n_a^1\times m_1} & \cdots & \Lambda_{n_a^1\times m_a} & \cdots & \Lambda_{n_a^1\times m_k} \end{bmatrix}$, in which the non-zero entries of $\Lambda_{n_a^1\times m_a}$ are positive and the entries of $\Lambda_{n_a^1\times m_s}$ for all $s \neq a$ are zeros. This is due to the fact that $\bar{P}_j^{aa} < \underline{P}_j^{ba}$, which indicates the followers are closer to the leaders in the same set $\mathcal{V}_a$ than the leaders in $\mathcal{V}_s$, $s \neq a$. Following similar argument, it can be shown that non-zero entries of $\Lambda_{n_a^s\times m}$, $s = 1, \ldots, \rho_a$, as well as the non-zero entries of Λ_a, $a = 1, \ldots, k$, are all positive.

By Lemma 1, we need to show the existence of a vector $\delta = \begin{bmatrix} \delta_1 & \ldots & \delta_{nm} \end{bmatrix}^T \in \mathbb{R}^{nm}$ such that $k = C\delta \in \mathbb{R}^n$ is element-wise positive. Due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \ldots m$, such that the first m entries of k are positive. Based on the analysis above, there also exist proper design of $\delta_i \in \mathbb{R}^+, i = m+1, \ldots mn$, such that the rest entries of k are positive. Hence, the system in (1) is herdable. ■

In Theorem 3, $\bar{P}_j^{aa}$ denotes the longest distance from the follower v_j in $\mathcal{V}_a$ to its own leader group in $\mathcal{V}_a$, and $\underline{P}_j^{ba}$ denotes shortest distance from the follower v_j in $\mathcal{V}_a$ to the leader group in $\mathcal{V}_a$. Thus, Theorem 3 indicates that the system in (1) is herdable by the selected leaders, if within any partitioned set the follower is closer (in terms of graph walks) to the leaders in the same partitioned set than to the leaders in other partitioned sets. To illustrate the developed

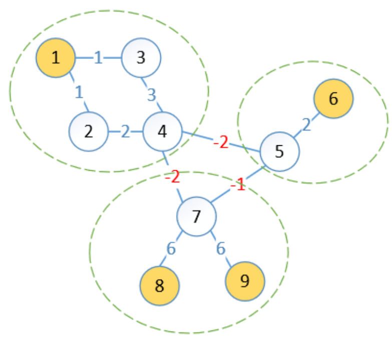

Figure 3. An example of leader group selection for herdability of weakly balanced signed graph, where the selected leaders are marked as solid nodes.

leader selection rule in Theorem 3, the following example is provided.

Example 3. Consider a weakly balanced signed network $\mathcal{G}$ with the partitioned sets $\mathcal{V}_1 = \{v_1, v_2, v_3, v_4\}$, $\mathcal{V}_2 = \{v_5, v_6\}$ and $\mathcal{V}_3 = \{v_7, v_8, v_9\}$. Following the leader selection rules in Theorem 3, Fig. 3 shows a herdable network where the leader group is selected as $\mathcal{V}_l = \{v_1, v_6, v_8, v_9\}$. It can be verified that the followers in $\mathcal{V}_i$, $i \in \{1, 2, 3\}$, is closer to the leaders in $\mathcal{V}_i$ than to the leaders in $\mathcal{V}_j$, $j \neq i$.

V. Conclusion

This work investigates leader selection algorithms for the herdability of structurally balanced signed graphs. Future research will consider extending current results to more general signed networks, such as structurally unbalanced graphs. Additional research will also consider optimal leader group selection, e.g., minimizing the size of the leader group.

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